JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **164**, 337–349 (1992)

On the Ensemble Average in the Study of Approximate Inertial Manifolds, II

DONGHO CHAE*

The Lefshetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

Submitted by C. Foias

Received September 18, 1990

In this note we obtained a new time averaged estimate of the distance between the orbit generated by solutions of 2-D Navier–Stokes equations and one of its approximate inertial manifolds in the space periodic case, using the Gevrey class regularity of the solutions in the space periodic case. This estimate is a substantial improvement of the author's previous one in the "very" high wave modes, or, physically, far in the dissipation range of turbulence. © 1992 Academic Press, Inc.

1. INTRODUCTION

For constructive approximation of the long time behavior of the solutions of the Navier–Stokes equations, Foias–Manely–Temam [4] introduced the notion of approximate inertial manifolds for the equation. Since then, its consequence on developing the new numerical methods [6], its connection to the classical physical theories of turbulence [3] as well as further theoretical investigation of the notion itself [7] have been studied. In particular, its relevance for the quasistatic behavior of the solutions was studied via time averaging in [3]. However, one fundamental problem, especially for physical applications, is to justify that the notion of approximate inertial manifold is really good approximation in physically relevant wave modes, i.e., wave numbers corresponding to the inertial range of turbulence. More specifically, the problem is to obtain a good estimate for the distance between an approximate inertial manifold and the orbit generated by the solutions of the equations. Even the combination of both analysis and heuristics made in [3] for the quasistatic study of 2-D Navier–Stokes equation lead only to the conclusion that the time-averaged distance estimate is "good" far in the dissipation range. In the author's previous paper [1] we obtained similar estimates, but without any heuristics.

* Current address: Department of Mathematics, Pohang Institute of Science and Technology, P.O. Box 125, Pohang, Kyung-Pook, Korea 790-600.

0022-247X/92 $3.00

In the present note we obtained a new estimate for the time-averaged distance, which improves the author's previous one substantially in the very far in the dissipation range. The crucial fact we used in the proof is the Gevrey class regularity for the solutions of the Navier–Stokes equations in the space-periodic domain. Unfortunately we did not yet succeed in extrapolating our estimates to the inertial range, the problem which should be further studied. In the next section after brief review of the notations we state our main propositions; these will be proved in the last section.

2. Notations and The Main Results

We consider the incompressible viscous fluid flow with viscosity $\nu > 0$ stirred by external force f, contained in a domain $\Omega = (0, 2\pi)^2$. As is well known (see, e.g., [2] or [9]) the Navier–Stokes equation describing the velocity field in this case can be written as an abstract evolution equation for u,

$$\frac{\partial u}{\partial t} + \nu A u + B(u) = f \tag{1}$$

in a Hilbert space H which consists of soleniodal vector fields in $L^2(\Omega)^2$ (with scalar product and norm denoted by $(\cdot, \cdot)$, $|\cdot|$). The Stokes operator, A, is a linear self adjoint unbounded positive operator with domain $D(A) \subset H$. Next, $B(u) = B(u, u)$, where

$$(B(u, v), w) = \sum_{j,k=1}^{2} \int_\Omega u_j \frac{\partial u_k}{\partial x_j} w_k \, dx.$$

We assume the space periodicity (with period $(0, 2\pi)^2$) for u and f, and moreover, for simplicity, we assume $\int_\Omega u(x, t)\, dx = 0$, $\forall t$. In this case we have the Fourier series expasion for each $u \in H$

$$u = \sum_{j \in \mathbf{Z}^2} u_j e^{ij \cdot x}, \qquad u_j \in \mathrm{C}^2, \quad u_{-j} = \bar{u}_j, \quad u_0 = 0 \tag{2}$$

with

$$j \cdot u_j = 0 \tag{3}$$

$$\sum_{j \in \mathbf{Z}^2} |u_j|^2 = \frac{1}{(2\pi)^2} |u|^2 < \infty. \tag{4}$$

For each $\alpha > 0$ the domain of A^α, $D(A^\alpha)$ is the set of functions satisfying (2)-(4) such that

$$(2\pi)^2 \sum_{j \in \mathbf{Z}^2} |j|^{4\alpha} |u_j|^2 = |A^\alpha u|^2 < \infty.$$

For the scalar product and the norm in $D(A^{1/2})$ we will use $((\cdot,\cdot))$, $\|\cdot\|$ for notation. For $\tau, s>0$ given we also consider the Gevrey class $D(e^{\tau A^s})$ that is a set of functions u satisfing (2)–(4) and

$$(2\pi)^2 \sum_{j\in\mathbf{Z}^2} e^{2\tau|j|^{2s}}|u_j|^2 = |e^{\tau A^s}u| < \infty.$$

In particular, for the scalar product and the norm in $D(e^{\tau A^{1/2}})$ we use $(\cdot,\cdot)_\tau$, $|\cdot|_\tau$, while for $D(e^{\tau A^{1/2}})$ we use $((\cdot,\cdot))_\tau$, $\|\cdot\|_\tau$ As before [1, 3] we decompuse u into two parts

$$u = y_m + z_m,$$

where

$$y_m = P_m u = \sum_{|j|\leqslant m} u_j(t)\, e^{ij\cdot x} \qquad \text{and} \qquad z_m = Q_m u = \sum_{|j|>m} u_j(t)\, e^{ij\cdot x}$$

characterizing the large and small scale structures of the flow, respecitively. In the above, m is any positive integer, and will be dropped in the subscript hereafter. Also, in the following we assume the stirring force f has only large scale component, i.e., $P_m f = f$; we note that, in particular, this implies $f\in D(e^{\sigma A^{1/2}})$ for any $\sigma>0$. By operating projectors P and Q on (1) we obtain

$$\frac{\partial y}{\partial t} + \nu A y + PB(y+z, y+z) = f \tag{5}$$

$$\frac{\partial z}{\partial t} + \nu A z + QB(y+z, y+z) = 0. \tag{6}$$

By $\{\lambda_j\}$ we denote the eigenvalues of A with

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to \infty \qquad \text{as} \quad j\to\infty.$$

The important parameters that will be used in the following are

$$\delta = \frac{\lambda_1}{\lambda_{m+1}}, \qquad L = \left(1 + \log\frac{\lambda_{m+1}}{\lambda_1}\right).$$

Two dimensionless numbers representing the degree of turbulence are the Grashof number G and the Reynolds number R, defined respectively by

$$G = \frac{|f|}{\nu^2\lambda_1}, \qquad R = \frac{\overline{(|u|^2)^{1/2}}}{\nu},$$

where the overbar denotes the long time average. In particular, in this note we use the long time average, defined by

$$\overline{(\cdot)} = \overline{\lim_{T \to \infty}} \frac{1}{T} \int_0^T (\cdot)\, dt.$$

We remark in passing that we do not know yet if we can replace the $\overline{\lim}$ by lim in the above definition, thus using the usual sense of long time average in our results; the problem is that we do not yet know if the limit exists or not for our integrands. However, we could replace the lim by LIM, the generalized (Banach) limit, as was done in [1]. For the initial conditions we assume $u_0 \in H$ as usual in the 2-D periodic case; this, combined with the assumption on the forcing f, entitles us to use the results in [5] on the Gevrey class regularity of the solution to (1); these results, including some arguments leading to them, will be briefly reviewed and refined into useful form for us in the proof of Lemma 1 of Section 3. We remark $G \geqslant R$, which is easy to check directly from (1). Finally let us recall that the approximate inertal manifold of our concern is the manifold $\mathcal{M}_1$, defined as the graph in H generated by the function

$$z = \Phi_1(y) \equiv -(\nu A)^{-1} QB(y, y).$$

(Recall we have assumed $Qf = 0$.) Our first main result is the following:

THEOREM 1. *There are constants C_1 and C_2 without dimension satisfying the following inequality*

$$\frac{\overline{\operatorname{dist}(u, \mathcal{M}_1)^2}}{\overline{|u|^2}} \leqslant C_1 R^{-2} G^7 \log G \exp\left(-\frac{C_2}{G^2 \log G\delta^{1/2}}\right) \delta^3 L. \tag{7}$$

We recall our previous estimate

$$\frac{\overline{\operatorname{dist}(u, \mathcal{M}_1)^2}}{\overline{|u|^2}} \leqslant C_1' R^{-1} G^4 \delta^3 L. \tag{8}$$

Using $R \leqslant G$, by simple calculation it is easy to verify that if

$$\delta^{1/2} G^2 \ll 1 \tag{9}$$

then (7) is an improved estimate of (8); on the other hand, this condition implies that we are in the very small scales of flow (very far in the dissipation range). As we shall see in the next section, Theorem 1 is a corollary the following lemma and the two propositions below.

LEMMA 1. *There exists a nondimensional constant C_3 satisfying the estimate*

$$\overline{|Au|_\sigma^2} \leqslant C_3 \nu^2 \lambda_1^2 G^4 \log G, \tag{10}$$

where σ is a constant defined in the proof.

For comparison we recall the inequality in [1]

$$\overline{|Au|^2} \leqslant \nu^2 \lambda_1^2 G^2. \tag{11}$$

As we will see in the proof the increased power of G in (10) compared to (11) results from the loss of orthogonality between Au and $B(u)$ in the scalar product $(\cdot, \cdot)_\sigma$.

PROPOSITION 1. *There are nondimensional constants C_4–C_6 of order unity satisfying the estimate*

$$\overline{|Az|_\sigma^2} \leqslant \begin{cases} C_4 G^2 \delta L\, \overline{|Au|_\sigma^2} & \text{in general} \\ C_6 G^2 \delta L\, \overline{|Ay|_\sigma^2} & \text{if } \ G^2 \delta L \leqslant C_5. \end{cases} \tag{12}$$

Since we have

$$\frac{\overline{|Az|^2}}{\overline{|Ay|^2}} \leqslant \frac{\overline{|Az|_\sigma^2}}{\overline{|Ay|_\sigma^2}}$$

we see that the second inequality in (12) is a sharper version of our previous one in [1],

$$\overline{|Az|^2} \leqslant C_4' G^2 \delta L\, \overline{|Ay|^2}. \tag{13}$$

We note, however, that (13) holds without any assumption on the size of $G^2 \delta L$.

PROPOSITION 2. *There exist dimensionless constants C_7–C_9 such that the following estimates are valid.*

$$|\nu\, \overline{|Az|_\sigma^2} + \overline{(Az, QB(y, y))_\sigma}| \leqslant C_7 \nu G (\delta L)^{1/2}\, \overline{|Az|_\sigma^2} \tag{14}$$

$$\begin{aligned} &|\, \overline{|QB(y, y)|_\sigma^2} + \nu \overline{(Az, QB(y, y))_\sigma}| \\ &\quad \leqslant \begin{cases} C_8 \nu^2 G^2 (\delta L)^{1/2} (\overline{|Au|_\sigma^2})^{1/2} (\overline{|Az|_\sigma^2})^{1/2} & \text{in general} \\ C_9 \nu^2 G^2 (\delta L)^{1/2} (\overline{|Ay|_\sigma^2})^{1/2} (\overline{|Az|_\sigma^2})^{1/2} & \text{if } \ G^2 \delta L \leqslant C_5, \end{cases} \end{aligned} \tag{15}$$

where C_5 is the same constant as in (12).

We observe that the two estimates in the above proposition have similar form to our previous ones in [1] except that the norm and the scalar product have been replaced from $(\cdot,\cdot)$, $|\cdot|$ to $(\cdot,\cdot)_\sigma$, $|\cdot|_\sigma$.

3. Proof of the Main Results

Proof of Lemma 1. We first refine the results in [5] in a form useful for our purpose; especially, we make every estimate independent of physical dimensions.

Let us put $\phi=\phi(t)=\nu\lambda_1^{1/2}\min\{t,\sigma_1\}$ for some constant σ_1. Taking the scalar product of (1) with Au in $D(e^{\phi A^{1/2}})$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_\phi^2+\nu\,|Au|_\phi^2\leqslant|(B(u,u),Au)_\phi|+\nu\lambda_1^{1/2}\,|(Au,A^{1/2}u)_\phi|+|(f,Au)_\phi|. \quad (16)$$

From [5, (2.26)],

$$|(B(u,u),Au)_\phi|\leqslant c_1\|u\|_\phi^2\,\|Au\|_\phi\left(1+\log\frac{|Au|_\phi^2}{\lambda_1\|u\|_\phi^2}\right)^{1/2}.$$

On the other hand, we have (see [4, (A.7)–(A.8)])

$$2c_1\,\|u\|_\sigma^2\,|Au|_\phi\left(1+\log\frac{|Au|_\phi^2}{\lambda_1\|u\|_\phi^2}\right)^{1/2}\leqslant\frac{\nu}{2}\,|Au|_\phi^2+\frac{c_1^2}{2\nu}\,\|u\|_\phi^4\log\frac{4c_1^2\,\|u\|_\phi^2}{\lambda_1\nu^2}. \quad (17)$$

And, from the Young's inequality we have

$$\nu\lambda_1^{1/2}\,|(Au,A^{1/2}u)_\phi|\leqslant\frac{\nu}{4}\,|Au|_\phi^2+\nu\lambda_1\,\|u\|_\phi^2 \quad (18)$$

and

$$|(f,Au)_\phi|\leqslant\frac{\nu}{4}\,|Au|_\phi^2+\frac{1}{\nu}\,|f|_\phi^2. \quad (19)$$

Therefore, we obtain from (16)–(19)

$$\frac{d}{dt}\|u\|_\phi^2+\frac{\nu}{2}\,|Au|_\phi^2\leqslant2\nu\lambda_1\,\|u\|_\phi^2+\frac{1}{\nu}\,|f|_\phi^2+\frac{c_1^2}{2\nu}\,\|u\|_\phi^4\log\frac{4c_1^2\,\|u\|_\phi^2}{\lambda_1\nu^2}. \quad (20)$$

Now, put

$$y=\frac{2+4c_1^2}{\lambda_1\nu^2}\,(|f|_\phi+\|u\|_\phi^2)$$

then, (20) implies

$$y' \leqslant \nu\lambda_1 c_2 y^2 \log y,$$

where c_2 is an appropriate nondimensional constant. Thus, as long as $y(t) \leqslant 2y_0 = 2y(0)$, we have

$$y' \leqslant c_2 \nu\lambda_1 y^2 \log 2y_0.$$

Solving this differential inequality we obtain

$$y \leqslant \frac{y_0}{1 - c_2\lambda_1 \nu t \log 2y_0}$$

and this is indeed $\leqslant 2y_0$ as long as

$$0 \leqslant t \leqslant T_*(\|u_0\|),$$

where we put

$$T_*(\|u_0\|) = \frac{c_3 \nu}{(|f| + \|u_0\|^2) \log(c_4/\lambda_1 \nu^2)(|f| + \|u_0\|^2)}$$

for suitable nondimensional constants c_3, c_4. Thus

$$|A^{1/2} e^{\phi(t) A^{1/2}} u(t)|^2 \leqslant 2(|f| + \|u_0\|^2)$$

for $0 \leqslant t \leqslant T_*(\|u_0\|)$. Actually we can further argue that $t \mapsto u(t)$ is analytic from $[0, \infty)$ into $D(A^{1/2} e^{\sigma A^{1/2}})$ for some σ a was done in [5]; we do not repeat that here. Now shifting the origin of time to $t_0 > 0$ and repeating the above argument, we obtain

$$|A^{1/2} e^{\phi(t) A^{1/2}} u(t + t_0)|^2 \leqslant 2(|f| + \|u(t_0)\|^2) \tag{21}$$

for $0 \leqslant t \leqslant T_*(\|u(t_0)\|)$, where

$$T_*(\|u(t_0)\|) = \frac{c_3 \nu}{(|f| + \|u(t_0)\|^2) \log(c_4/\lambda_1 \nu^2)(|f| + \|u(t_0)\|^2)}. \tag{22}$$

Let $t_0 \geqslant t_* =$ entering time into the absorbing set [8]

$$\left\{ \|u\|^2 \leqslant \frac{2\,|f|^2}{\nu^2 \lambda_1} = 2G^2 \nu^2 \lambda_1 \right\}.$$

Then, elementary computation after substituting $|f| = G\nu^2\lambda_1$, $\|u(t_0)\|^2 \leqslant 2G^2\nu^2\lambda_1$ into (22) leads to

$$T_*(\|u(t_0)\|) \geqslant \frac{c_5}{\nu\lambda_1 G^2 \log G}$$

for $t_0 \geqslant t_*$, where c_5 is a nondimensional constant. We now set

$$T_1 = \frac{c_5}{\nu\lambda_1 G^2 \log G}. \tag{23}$$

Then, for $t_0 \geqslant t_*$, $0 \leqslant t \leqslant T_1$ we have from (21),

$$\begin{aligned} |A^{1/2}e^{\phi(t)A^{1/2}}u(t+t_0)|^2 &\leqslant 2(|f| + 2G^2\nu^2\lambda_1) \\ &\leqslant 2(G\nu^2\lambda_1 + 2G^2\nu^2\lambda_1) \leqslant 6G^2\nu^2\lambda_1. \end{aligned}$$

Especially,

$$|A^{1/2}e^{\phi(T_1)A^{1/2}}u(T_1+t_0)|^2 \leqslant 6G^2\nu^2\lambda_1.$$

Thus, if we put $T_2 = t_* + T_1$, $\sigma = \phi(T_1)$, then

$$\sup_{T_2 \leqslant t < \infty} \|u\|_\sigma^2 \leqslant 6G^2\nu^2\lambda_1. \tag{24}$$

Now, by the similar procedure leading to (20) we obtain from (1)

$$\frac{d}{dt}\|u\|_\sigma^2 + \nu\,|Au|_\sigma^2 \leqslant \frac{1}{\nu}|f|_\sigma^2 + \frac{c_1}{\nu}\|u\|_\sigma^4 \log\frac{4c_1^2\,\|u\|_\sigma^2}{\lambda_1\nu^2}. \tag{25}$$

Taking the long time average of (25), then the term involving the time derivative vanishes, and noting

$$\bar{a} = \overline{\lim_{T\to\infty}}\,\frac{1}{T}\int_{T_2}^{T} a\,dt \qquad \text{for} \quad a \in L^1(0, T_2) \tag{26}$$

we obtain from (24)

$$\begin{aligned} \nu\,\overline{|Au|_\sigma^2} &\leqslant \frac{1}{\nu}|f|_\sigma^2 + \sup_{t \geqslant T_2}\frac{c_1^2}{\nu}\|u\|_\sigma^4 \log\frac{4c_1^2\,\|u\|_\sigma^2}{\lambda_1\nu^2} \\ &\leqslant \frac{1}{\nu}e^{2\sigma\lambda_1^{1/2}}|f|^2 + c_6\nu^3\lambda_1^2 G^4 \log G \end{aligned} \tag{27}$$

for an appropriate nondimensional constant c_6. For the forcing term we have, for $G \geqslant 1$,

$$e^{2\sigma\lambda_1^{1/2}} |f|^2 = e^{2\lambda_1 \nu \min\{\sigma_1, T_1\}} |f|^2 \leqslant e^{2\lambda_1 \nu T_1} |f|^2$$
$$= e^{2(c_5/G^2 \log G} G^2 \nu 4\lambda_1^2 \leqslant c_7 G^2 \nu^4 \lambda_1^2,$$

where c_7 is another dimensionless constant. This, combined with (27) leads to (10), thus completing the proof of Lemma 1.

Before proving Propositions 2, 3 we recall some useful facts. First, as an immediate corollary of Lemma 1 (especially its proof) and its following remark in [5] we observe that the estimates of $|B(u, v)|_\sigma$ are in the exactly same forms as $|B(u, v)|$ with the norms $|\cdot|$, $\|\cdot\|$ replaced by $|\cdot|_\sigma$, $\|\cdot\|_\sigma$, respectively. For example, as an analogue of [1, (15)–(18)] we have

$$|B(u, v)|_\sigma \leqslant \begin{cases} c_8 \, |u|_\sigma^{1/2} \, |Au|_\sigma^{1/2} \, \|v\|_\sigma \\ c_1 \, \|u\|_\sigma \, \|v\|_\sigma \, (1 + \log(|Au|_\sigma^2/\lambda_1 \, \|u\|_\sigma^2))^{1/2} \\ c_1 \, |u|_\sigma \, |A_v|_\sigma \, (1 + \log(|Av|_\sigma^2/\lambda_1 \, \|v\|_\sigma^2))^{1/2}. \end{cases} \tag{28}$$

We also recall that as a consequence of [1, (13)–(14)]

$$|Ay|_\sigma \leqslant \lambda_{m+1}^{1/2} \, \|y\|_\sigma \leqslant \lambda_{m+1} \, |y|_\sigma, \qquad \forall y \in PH \tag{29}$$

$$|z|_\sigma \leqslant \lambda_{m+1}^{-1/2} \, \|z\|_\sigma \leqslant \lambda_{m+1}^{-1} \, |Az|_\sigma \qquad \forall z \in QH. \tag{30}$$

Combining (28)–(30) we have an analogue of [1, Lemma 1]:

LEMMA 2. *The following estimates are valid for any $y \in PH$ and $z \in QH$;*

$$|B(y, y)|_\sigma \leqslant c_1 L^{1/2} \, \|y\|_\sigma^2 \tag{31}$$

$$|B(y, z)|_\sigma \leqslant \begin{cases} c_8 \lambda_1^{-1} \delta^{1/2} \, |Ay|_\sigma \, |Az|_\sigma \\ c_1 \lambda_1^{-1/2} (\delta L)^{1/2} \, \|y\|_\sigma \, |Az|_\sigma \end{cases} \tag{32}$$

$$|B(z, y)|_\sigma \leqslant c_8 \lambda_1^{-1/2} \delta^{1/2} \, \|y\|_\sigma \, |Az|_\sigma \tag{33}$$

$$|B(z, z)|_\sigma \leqslant c_8 \lambda_2^{-1/2} \delta^{1/2} \, \|u\|_\sigma \, |Az|_\sigma. \tag{34}$$

Finally we recall an algebraic identity in [1],

$$AB(u, u) = B(u, Au) - B(Au, u). \tag{35}$$

We are now ready to prove Proposition 2.

Proof of Proposition 1. Taking scalar product of (6) with Az in $(\cdot, \cdot)_\sigma$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|_\sigma^2 + \nu\,|Az|_\sigma^2 = (Az, QB(y, y))_\sigma + (Az, QB(y, z))_\sigma$$
$$+ (Az, QB(z, y))_\sigma + (Az, QB(z, z))_\sigma. \tag{36}$$

We now estimate pointwise in time for $t \geqslant T_2$ each term of the right hand side of (36). The common procedure of the estimates is the following; first we use Hoelder estimate for the scalar product, and then use Lemma 4 for the estimate of B, next, using (29) – (30) to obtain terms involving $\|y\|_\sigma$, $\|z\|_\sigma$, and $|Az|_\sigma$, finally we use the uniform in time estimate for $\|y\|_\sigma$, $\|z\|_\sigma$ ($\leqslant \|u\|_\sigma$) in (24). This leads to

$$\begin{aligned}
|(Az, QB(y, y))_\sigma| &\leqslant |Az|_\sigma\,|QB(y, y)|_\sigma \\
&\leqslant \lambda_{m+1}^{-1}\,|Az|_\sigma\,|AQB(y, y)|_\sigma \\
&\leqslant \lambda_{m+1}^{-1}\,|Az|_\sigma\,(|B(y, Ay)|_\sigma + |B(Ay, y)|_\sigma) \\
&\leqslant 2c_1 L^{1/2}\lambda_{m+1}^{-1}\,|Az|_\sigma\,\|y\|_\sigma\,|Ay|_\sigma \\
&\leqslant 2\sqrt{6}\,c_1 G\nu(\delta L)^{1/2}\,|Az|_\sigma\,|Ay|_\sigma \\
&\leqslant \frac{\nu}{2}\,|Az|_\sigma^2 + 12c_1^2\nu G^2\,\delta L\,|Ay|_\sigma^2,
\end{aligned} \tag{37}$$

where we used $AQ = QA$ and (35) in the third step.

$$\begin{aligned}
|(Az, QB(y, z))_\sigma| &\leqslant |Az|_\sigma\,|B(y, z)|_\sigma \\
&\leqslant c_1\lambda_1^{-1/2}(\delta L)^{1/2}\,|Az|_\sigma^2\,\|y\|_\sigma \\
&\leqslant \sqrt{6}\,c_1\nu G(\delta L)^{1/2}\,|Az|_\sigma^2
\end{aligned} \tag{38}$$

$$\begin{aligned}
|(Az, QB(z, y))_\sigma| &\leqslant |Az|_\sigma\,|B(z, y)|_\sigma \\
&\leqslant c_8\lambda_1^{-1/2}\delta^{1/2}\,|Az|_\sigma^2\,\|y\|_\sigma \\
&\leqslant \sqrt{6}\,c_8\nu G\delta^{1/2}\,|Az|_\sigma^2
\end{aligned} \tag{39}$$

$$\begin{aligned}
|(Az, QB(z, z))_\sigma| &\leqslant |Az|_\sigma\,|B(z, z)|_\sigma \\
&\leqslant c_8\lambda_1^{-1/2}\delta^{1/2}\,|Az|_\sigma^2\,\|u\|_\sigma \\
&\leqslant \sqrt{6}\,\mathrm{c}8\nu\delta^{1/2}G\,|Az|_\sigma^2.
\end{aligned} \tag{40}$$

Combining (37)–(39) with (36) we obtain the inequality

$$\frac{d}{dt}\|z\|_\sigma^2 + \nu\,|Az|_\sigma^2 \leqslant \sqrt{6}(c_1 + 2c_8)\,\nu G(\delta L)^{1/2}\,|Az|_\sigma^2 + 12c_1^2\nu G^2\delta L\,|Ay|_\sigma^2. \tag{41}$$

Taking the long time average the time derivative term of (41) vanishes as before, thus

$$\nu\,\overline{|Az|_\sigma^2} \leqslant \sqrt{6}(c_1+2c_8)\,\nu G(\delta L)^{1/2}\,\overline{|Az|_\sigma^2} + 12c_1^2 \nu G^2\,\delta L\,\overline{|Ay|_\sigma^2}. \qquad (42)$$

From (42) we have, if $\sqrt{6}(c_1+2c_8)\,G(\delta L)^{1/2} \leqslant \frac{1}{2}$, then

$$\overline{|Az|_\sigma^2} \leqslant 24c_1^2 G^2\,\delta L\,\overline{|Ay|_\sigma^2}.$$

Thus, we have the second inequality of (12) for $C_5=(1/2\sqrt{6})(c_1+2c_8)^{-1}$, $C_6=24c_1$. Otherwise, from $|Az|_\sigma, |Ay|_\sigma \leqslant |Au|_\sigma$, and adding the two terms in the right hand side of (42) we obtain the first inequality of (12). This completes the proof of Proposition 1.

Proof of Proposition 2. (i) Proof of (14): We take scalar product (6) with Az in $(\cdot\,\cdot)_\sigma$, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|_\sigma^2 + \nu\,|Az|_\sigma^2 + (Az, QB(y, y))_\sigma$$
$$+ (Az, B(y, z) + B(z, y) + B(z, z))_\sigma = 0.$$

Thus, after the long time average (the time derivative term vanishes as before) we have

$$|\nu\,\overline{|Az|_\sigma^2} + \overline{(Az, QB(y, y))_\sigma}|$$
$$\leqslant |\overline{(Az, QB(y, z))_\sigma}| + |\overline{(Az, QB(z, y))_\sigma}| + |\overline{Az, QB(z, z))_\sigma}| \qquad (43)$$

The estimates for the right hand side of (43) before the time average were already done in (38)–(40) in the proof of Proposition 1. Thus, taking time average and adding them together we obtain (14).

(ii) Proof of (15): Taking the scalar product of (6) with $QB(y, y)$ in $(\cdot, \cdot)_\sigma$ we have

$$|QB(y, y)|_\sigma^2 + \nu(Az, QB(y, y))_\sigma$$
$$= -\frac{d}{dt}(z, B(y, y))_\sigma + (z, B(y', y) + B(y, y'))_\sigma$$
$$+ (QB(y, y), B(y, z) + B(z, y) + B(z, z))_\sigma. \qquad (44)$$

Since

$$|(z, B(y, y))_\sigma| \leqslant c_1 L^{1/2}\,|z|_\sigma\,\|y\|_\sigma^2 \leqslant c_1 L^{1/2}\lambda_{m+1}^{-1/2}\,\|u\|_\sigma^3 < \infty$$

for all $t \geqslant T_2$ in virtue of (26), we infer

$$\overline{\left|\frac{d}{dt}(z, B(y, y))_\sigma\right|} = 0.$$

Thus, after substituting

$$y' = -\nu A y - PB(y+z, y+z) = f$$

into (44), and taking the time average of it we obtain

$$\begin{aligned}
&|\,\overline{|QB(y, y)|_\sigma^2} + \nu\overline{(Az, QB(y, y))_\sigma}\,| \\
&\quad \leqslant \nu\,\overline{|(z, B(Ay, y))_\sigma|} + \nu\,\overline{|(z, B(y, Ay))_\sigma|} \\
&\qquad + \overline{|(z, B(PB(y+z, y+z), y))_\sigma|} + \overline{|(z, B(y, PB(y+z, y+z)))_\sigma|} \\
&\qquad + \overline{|(QB(y, z) + QB(z, y) + QB(z, z), B(y, y))_\sigma|} \\
&\qquad + \overline{|(z, B(f, y))_\sigma|} + \overline{|(z, B(y, f))_\sigma|}.
\end{aligned} \tag{45}$$

Now the estimate of the right hand side of (45), following the procedure described in the proof of Proposition 1, leading to (15) is essentially repetition of [1, Proof of Theorem 3] with trivial modifications, which we omit here.

Proof of Theorem 1. First, we have

$$\begin{aligned}
\frac{\overline{\operatorname{dist}(u, \mathcal{M}_1)^2}}{\overline{|u|^2}} &\leqslant (\overline{|u|^2})^{-1}\, \overline{|z + (\nu A)^{-1} QB(y, y)|^2} \\
&\leqslant (\overline{|u|^2})^{-1} (\nu\lambda_{m+1})^{-2} \exp(-\sigma\lambda_{m+1}^{1/2})\, \overline{|\nu Az + QB(y, y)|_\sigma^2}.
\end{aligned}$$

Next, by sequence of use of Proposition 2, Proposition 1, and Lemma 1 we have

$$\begin{aligned}
\overline{|\nu Az + QB(y, y)|_\sigma^2} &\leqslant |\nu^2\, \overline{|Az|_\sigma^2} + \nu\overline{(Az, QB(y, y))_\sigma}| \\
&\quad + |\,\overline{|QB(y, y)|_\sigma^2} + \nu\overline{(Az, QB(y, y))_\sigma}| \\
&\leqslant (C_4^{1/2} C_8 + C_9)\, G^2 \nu^2 (\delta L)^{1/2} (\overline{|Au|_\sigma^2})^{1/2} (\overline{|Az|_\sigma^2})^{1/2} \\
&\leqslant C_4^{1/2} (C_4^{1/2} C_8 + C_9)\, \nu^2 G^3\, \delta L\, \overline{|Au|_\sigma^2} \\
&\leqslant C_3 C_4^{1/2} (C_4^{1/2} + C_9)\, \nu^4 \lambda_1^2 G^7 \log G\, \delta L.
\end{aligned}$$

Thus, combining the above two estimates, we obtain

$$\frac{\overline{\operatorname{dist}(u, \mathcal{M}_1)^2}}{\overline{|u|^2}} \leqslant cR^{-2} G^7 \log G \exp(-\nu(\lambda_1 \lambda_{m+1})^{1/2} \min\{T_1, \sigma_1\})\, \delta^3 L. \tag{46}$$

Choosing $\sigma_1 = T_1$ in the above we obtain (7) from (23). This completes the proof of Theorem 1.

ACKNOWLEDGMENTS

The author thanks Professor Ciprian Foias for suggesting the problem and for encouragements. This research was partially supported by NSF Grant DMS-8802596 through the Institute for Applied Mathematics and Scientific Computing at the Indiana University-Bloomington and by the research fund of Indiana University.

REFERENCES

1. D. CHAE, On the ensemble average in the study of approximate inertial manifolds, *J. Math. Anal. Appl.*, to appear.
2. P. CONSTANTIN, AND C. FOIAS, Navier–Stokes equations, *in* "Chicago Lectures in Math. Series," Univ. of Chicago Press, Chicago, 1988.
3. C. FOIAS, O. P. MANLEY, AND R. TEMAM, Approximate inertial manifolds and effective viscosity in turbulent flows, to appear.
4. C. FOIAS, O. P. MANLEY, AND R. TEMAM, Modelling of small and large eddies in two dimensional turbulent flows, *in* "Math. Modelling and Num. Anal. Vol. 22, No. 1, pp. 93–114, 1988.
5. C. FOIAS AND R. TEMAM, Gevrey class regularity for the solutions of the Navier–Stokes equations, *J. Funct. Anal.* **87** (1989), 359–369.
6. M. MARION AND R. TEMAM, Nonlinear Galerkin methods, *SIAM J. Numer. Anal.* **26** (1989).
7. R. TEMAM, "Dynamical Systems, Turbulence and the Numerical Solution of the Navier–Stokes Equations," The Institute for Applied Mathematics & Scientific Computing, Indiana Univ.-Bloomington, preprint.
8. R. TEMAM, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.
9. R. TEMAM, Navier–Stokes equations and nonlinear functional analysis, *in* CBMS-NSF Regional Conference Series in Appl. Math., SIAM, 1983."